Public-key cryptography in the pre- and post-quantum world

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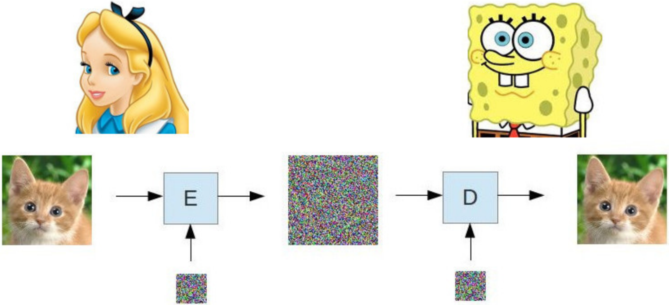
Catholic University of Lille

Symmetric cryptography



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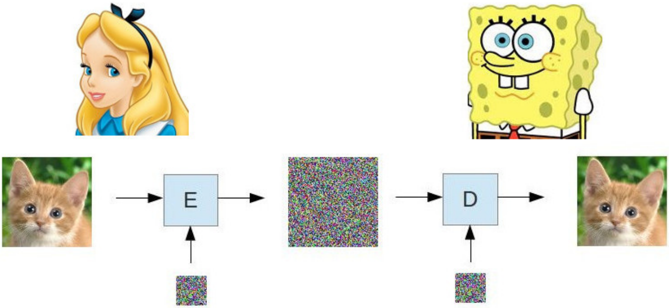

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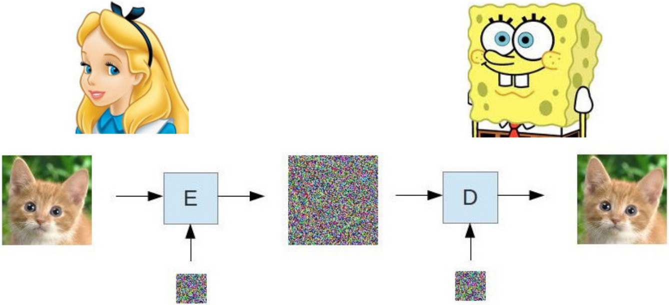

- encryption and decryption (knowing the key) should be fast
- decryption without the proper key should be **LONG**

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Advanced Encryption Standard

- A federal standard since 2001

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- Key distribution might be a problem...

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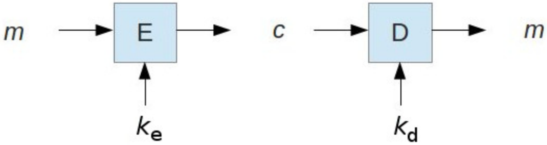

- k_e public, k_d private : public-key encryption
- k_e private, k_e public : digital signature

Asymmetric cryptography

A solution to the key distribution problem

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They could also digitally sign their exchanges to avoid man-in-the-middle attacks.

Modular arithmetic

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$$4 + 8 \cdot 2 = 4 + 16 \underset{12}{\equiv} 4 + 4 = 8$$

With a 24-hour clock :

$$16 + 8 \cdot 2 = 16 + 16 = 32 \underset{24}{\equiv} 8$$

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and then

$$b^{40} \underset{541}{\equiv} b^{32} \cdot b^8 \equiv 79 \cdot 221 \equiv 147.$$

Modular arithmetic

Taking powers

Much quicker than first computing

$$\begin{aligned} b^{40} = {} & 66146476117266938411 \\ & 57619437514125541994 \\ & 77293169155435203018 \\ & 62889699022438451615 \\ & 53331941376 \end{aligned}$$

then taking the remainder modulo 541 !

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$$\begin{aligned} b^{40} = {} & 661464761172669384115 \\ & 76194375141255419947 \\ & 72931691554352030186 \\ & 28896990224384516155 \\ & 3331941376 \end{aligned}$$

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Number of steps needed is proportional to $\log_2$ of the exponent.

Rivest-Shamir-Adleman (1977)

Given a (large) integer n :

$$\begin{cases} E(e, m) \underset{n}{\equiv} m^e \\ D(d, c) \underset{n}{\equiv} c^d \end{cases}$$

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If $n = p_1 \cdots p_\ell$, then $\phi(n) = (p_1 - 1) \cdots (p_\ell - 1)$.

RSA

A working example

Take

$$n = 3670486004008413084113 77,$$

$$m = 1010101010101010101010,$$

$$e = 3,$$

$$d = 2446990669330863306993 07.$$

RSA

A working example

Take

$$n = 36704860040084130841137 7,$$

$$m = 10101010101010101010101 0,$$

$$e = 3,$$

$$d = 24469906693308633069930 7.$$

Then Alice computes

$$c \underset{n}{\equiv} m^e \underset{n}{\equiv} 28017227544946476129772 7$$

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A working example

Take

$$n = 367048600400841308411377,$$

$$m = 101010101010101010101010,$$

$$e = 3,$$

$$d = 244699066933086330699307.$$

Then Alice computes

$$c \underset{n}{\equiv} m^e \underset{n}{\equiv} 280172275449464761297727$$

and Bob is able to decrypt this to

$$c^d \underset{n}{\equiv} 101010101010101010101010.$$

Knowing e (or d), it is easy to recover the other key in $\log(n)$ steps by using Euclid's algorithm backwards :

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Best currently known algorithm : first **factor** n as

$$n = p_1 \cdots p_\ell$$

then (trivially) compute

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RSA
Factoring n

Assume $n = p_1 \cdots p_\ell$ with $p_1 < \ldots < p_\ell$.

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Might as well take $\ell = 2$! Then $n = pq$ with p, q prime and

$$\phi(n) = (p-1)(q-1).$$

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$$L_\alpha(n) := \exp\left(c\,(\log n)^\alpha\,(\log\log n)^{1-\alpha}\right).$$

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- $L_0(n) = (\log n)^c$
- $L_1(n) = n^c$
- in general $L_\alpha(n)$ is somewhere between these two.

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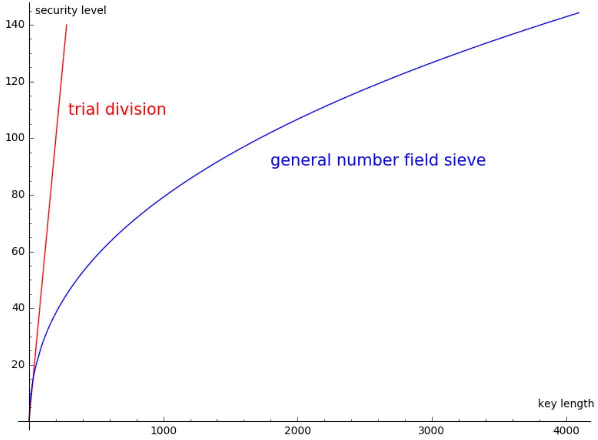
- General number field sieve (1993) : $L_{\frac{1}{3}}$ avec $d \approx 1{,}923$.

RSA

Equivalent symmetric ciphers



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Same algorithmic complexity than factoring (DSA, Diffie-Hellman)

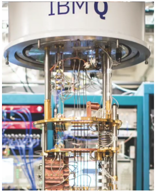
But : $\sqrt{n}$ in more general groups

$\implies$ methods based on elliptic curves (ECDSA, ECDH)

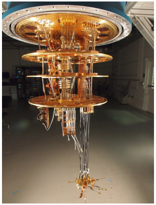
Quantum computers

State of the art

Simulated annealing



IBM : 50 qubits



Google : 70 qubits

Quantum computers

Grover's algorithm

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$\implies$ symmetric keys will need to be twice as long

Quantum computers

Shor's algorithm

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$$(a^{\frac{r}{2}})^2 \underset{p \cdot q}{\equiv} 1 \implies \begin{cases} a^{\frac{r}{2}} \underset{p}{\equiv} \pm 1 \\ a^{\frac{r}{2}} \underset{q}{\equiv} \pm 1 \end{cases}$$

There is a 50% chance that

$$\gcd(a^{\frac{r}{2}} - 1, n) \quad \text{and} \quad \gcd(a^{\frac{r}{2}} + 1, n)$$

are non-trivial factors of n.

Quantum computers

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Factors n in time

$$(\log n)^2(\log \log n)(\log \log \log n)$$

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~~RSA~~ ~~(EC)DSA~~ ~~(EC)DH~~

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Ongoing standardization process by NIST.

Consider a lattice $\Lambda = \{ \sum_i a_i \mathbf{v}_i \mid a_i \in \mathbf{Z} \}$ where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independant vectors in $\mathbf{R}^n$.

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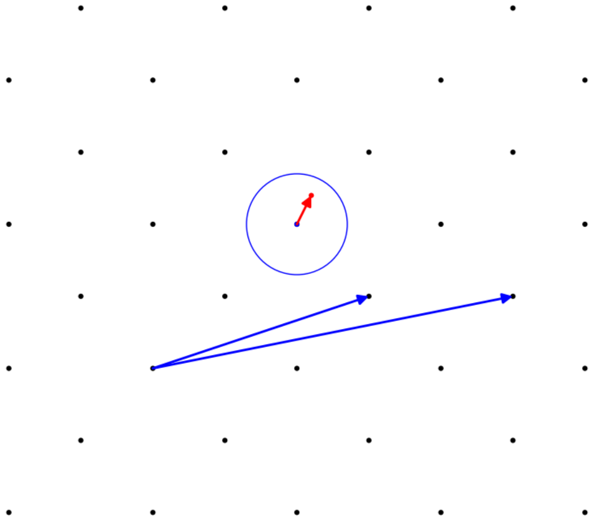
Algorithmically difficult problem if an adapted basis is not known.

Post-quantum encryption

Lattice-based encryption



Thanks !